PG – 220

II Semester M.Sc. Examination, June 2016
(CBCS)
MATHEMATICS
M 205 T : Functional Analysis

Time : 3 Hours　　　　　　　　　　　　　　　　　　　Max. Marks : 70

Instructions : *1) Answer* ***any five full*** *questions .*
2) ***All*** *questions carry* ***equal*** *marks.*

1. a) Define a norm and normed linear space. Show that in a normed linear space the following hold :

   i) Norm is continuous

   ii) Addition is jointly continuous. **6**

   b) Prove the following :

   i) Holder's inequality

   ii) Minkowskis inequality for l_p^n. **8**

2. a) If N and N′ are normed linear spaces over the same field and $T : N \to N'$ is a linear transformation then prove the following are equivalent :

   i) T is continuous

   ii) T is continuous at the origin.

   iii) T is bounded

   iv) T (s) is bounded in N′, where $s = \{x : \|x\| = 1\}$ is a closed unit ball in N. **8**

   b) Define Banach space. Let $(B, \| \ \|)$ be a Branch space such that $B = M \oplus N$, where M and N are linear subspaces of B. Then show that $\|z\|_1 = \|x\| + \|y\|$, for every $z = x + y \in B$ is a norm on B. **6**

3. a) Prove that there is a natural embedding of N into N** by the isometric isomorphism $\phi : N \to N^{**}$ defined by $\phi(x) = F_x$, for all x in N, $F_x(f) = f(x)$, for all f in N*. **8**

P.T.O.

b) If N is a normed linear space and $x_0 \in N$, $x_0 \neq 0$, then prove that there is a $f_0 \in N^*$ such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$ and hence deduce that N* separates vectors in N. **6**

4. a) State open mapping theorem. Prove closed graph theorem. **7**

b) State and prove uniform boundedness theorem. **7**

5. a) Define a Hilbert space. Prove that in a Hilbert space H the following hold

i) $|\langle x, y\rangle| \leq \|x\|\|y\|$

ii) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in H$ **7**

b) Define orthogonal complement of a set $S \subset H$. Prove the following :

i) $S \cap S^{\perp} \subset \{0\}$

ii) $S_1 \subset S_2 \Rightarrow S_2^{\perp} \subset S_1^{\perp}$

iii) $S^{\perp}$ is a closed subspace of H. **7**

6. a) Let $\{e_1, e_2, \ldots, e_n\}$ be a complete orthonormal set in a Hilbert space H. If $x \in H$ then prove the following :

i) $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2$

ii) $x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \perp e_j$, for each j. **5**

b) Define translation. Prove that translation preserves convexity and closure. **4**

c) If X is an inner product space and 'y' is some fixed vector in X, then prove that $f_y : X \to F : f_y(x) = \langle x, y\rangle, \forall x \in X$ is a linear functional on X. **5**

7. a) Define self adjoint operator. If A_1 and A_2 are self adjoint operators on H, then prove that $A_1 A_2$ is self adjoint if and only if $A_1 A_2 = A_2 A_1$. **4**

b) If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other, then prove that $N_1 + N_2$ and N_1N_2 are normal. **6**

c) Prove that an operator T is normal if and only if its real and imaginary parts commute. **4**

8. a) Define unitary operator. Prove that an operator T is unitary if and only if it is an isometric isomorphism of H onto itself. **7**

b) Show that an operator T on a Hilbert space H is unitary if and only if $\{e_i\}$ is a complete orthonormal set implies $\{Te_i\}$ is a complete orthonormal set. **7**